The "Mistake" Method: It's Similarity to the "ac" Method, and Understanding Why It Works

We consider here quadratic trinomials of the general form $ax^2 + bx + c$, where a, b, and c are integers. We are hoping to find two linear binomial factors, say $(px + m)$ and $(qx + n)$, where m, n, p, and q are all integers, such that

$$ax^2 + bx + c = (px + m)(qx + n). \qquad (1)$$

This is what it means to factor the trinomial. We must find the integers m, n, p, and q, given the numbers a, b, and c. Whatever these integers are, we can use the distributive rule (or, if you prefer, the "FOIL" algorithm) to multiply $(px + m)$ and $(qx + n)$ together to obtain

$$(px + m)(qx + n) = px(qx + n) + m(qx + n) = pqx^2 + pnx + qmx + mn,$$

or, collecting the two like terms involving x, we will have

$$(px + m)(qx + n) = pqx^2 + (mq + np)x + mn.$$

We want this to agree with the original trinomial $ax^2 + bx + c$, that is,

$$ax^2 + bx + c = pqx^2 + (mq + np)x + mn,$$

so we must be sure that the integers m, n, p, and q satisfy

$$pq = a, \quad mq + np = b, \quad \text{and} \quad mn = c. \qquad (2)$$

And that is how we begin to look for factors of the original trinomial. In particular, the product pq of the coefficients of x in the two factors has to equal a, and the product mn of the constants in the two factors has to equal c. Then we worry later about whether after making those choices we get agreement with the middle term: $mq + np = b$ (this is the "trial and error" method of factoring the trinomial).

Now, in the sum defining b of equation (2), the product mq is some new integer M, and the product np is another new integer N, so that $b = M + N$, the sum of these two integers. Notice that if we multiply a and c together, we get $ac = pq \cdot mn$, but since the order of multiplication makes no difference, we could also write it as $ac = mq \cdot np$, or $ac = MN$, the *product* of the same two integers that must *sum* to b! Instead of the four integers m, n, p, and q, we are thus led to search first for *just two* integers, M and N, whose product is the same as ac, and whose sum is b, that is:

$$MN = ac, \quad \text{and} \quad M + N = b. \qquad (3)$$

The "*ac*" method begins by seeking these two integers M and N. If two such integers are found, then b is replaced by $M + N$ in the original trinomial to write it as

$$ax^2 + bx + c = ax^2 + (M + N)x + c = ax^2 + Mx + Nx + c. \quad (\text{where } MN = ac, \text{ and } M + N = b) \qquad (4)$$

One then tries to factor the four terms on the right-hand side by grouping. This will work only if two pairs of coefficients have a common factor, for example, if a and M have a common numerical factor, and N and c have a common numerical factor, say $a = pq$ and $M = mq$, and $N = pn$ and $c = mn$. If this is the case, then the right-hand side can be rewritten to obtain

$$ax^2 + bx + c = pqx^2 + mqx + npx + mn.$$

Factoring qx from the first two terms, and n from the last two terms yields

$$ax^2 + bx + c = qx(px + m) + n(px + m),$$

and factoring the common factor of $(px + m)$ from the two remaining terms yields the factored result of equation (1):

$$ax^2 + bx + c = (px + m)(qx + n).$$

The so-called "mistake" method proceeds along the same lines by first finding the two integers M and N satisfying $MN = ac$, and $M + N = b$. The coefficient b is replaced by $b = M + N$, as with the "ac" method, but in addition, the relation $ac = MN$ is used to replace $c = MN/a$ in the original trinomial. Making both replacements, the trinomial can be rewritten as

$$ax^2 + bx + c \,=\, ax^2 + (M+N)x + \frac{MN}{a}.$$

The least common denominator on the right-hand side is a, which lets us write the trinomial as

$$ax^2 + bx + c \,=\, \frac{a^2x^2 + aMx + aNx + MN}{a},$$

after removing the parentheses in the numerator. The four terms of the numerator can again be factored by grouping. Factoring ax from the first two terms, and N from the last two terms yields

$$ax^2 + bx + c \,=\, \frac{ax(ax+M) + N(ax+M)}{a},$$

and factoring the common factor of $(ax + M)$ from the remaining two terms yields the general form

$$ax^2 + bx + c = \frac{(ax+M)(ax+N)}{a}. \quad (\text{where } MN = ac, \text{ and } M + N = b) \tag{5}$$

This is a perfectly fine factored form of the trinomial, although you may well be wondering what it has to do with the factored form, equation (1), we hoped to find. Again, it depends on the integers a and M having a common factor, and a and N having a common factor. For example, if $a = pq$ and $M = mq$, and N has a factor in common with a, say $N = np$, then equation (5) will reduce to

$$ax^2 + bx + c \,=\, \frac{(pqx+mq)(pqx+np)}{pq} \,=\, \frac{q(px+m)\cdot p(qx+n)}{pq} \,=\, (px+m)(qx+n),$$

where we factored q from the first binomial factor, and p from the second binomial factor, then cancelled the product pq in the numerator with the denominator. This is, of course, the desired result, the coefficient a in equation (5) "disappearing" from the final factorization.

Comment In effect, the "mistake" method (5) has the additional factoring-by-grouping step required by the "ac" method (4) incorporated in it, and all that is left is to complete the factoring of common factors that remain in the coefficients a, M, and N of the binomial factors to obtain the completely factored form (1).

SOME EXAMPLES

The following are a few examples of the use of the "mistake" method, i.e., equation (5):

1. $6x^2 - x - 2$.

   Here, $a = 6$, $b = -1$, and $c = -2$, so $ac = -12$. We seek M and N such that $MN = ac = -12$ and $M + N = b = -1$. Clearly, $M = -4$ and $N = 3$ satisfy both criteria, so equation (5) yields

   $$6x^2 - x - 2 \,=\, \frac{(6x-4)(6x+3)}{6}.$$

   Now, in the numerator, a common factor of 2 can be factored from the first binomial, and a common factor of 3 can be factored from the second binomial, to yield

   $$6x^2 - x - 2 \,=\, \frac{2(3x-2)\cdot 3(2x+1)}{6} \,=\, \frac{6(3x-2)(2x+1)}{6}.$$

   Cancelling the common factor of 6 in the numerator and denominator gives the completely factored form:

   $$6x^2 - x - 2 \,=\, (3x-2)(2x+1).$$

2. $2x^2 + 3x - 2$

We have $a = 2$, $b = 3$, and $c = -2$, so $ac = -4$. We seek M and N such that $MN = ac = -4$ and $M + N = b = 3$. Clearly, $M = 4$ and $N = -1$ satisfy both criteria, so from equation (5):

$$2x^2 + 3x - 2 = \frac{(2x+4)(2x-1)}{2} = \frac{2(x+2)(2x-1)}{2} = (x+2)(2x-1).$$

Here, a common factor of 2 is factored from the first binomial factor in the numerator, and this cancels the 2 in the denominator.

3. $4x^2 + 16x + 15$.

We have $a = 4$, $b = 16$, $c = 15$. So $ac = 4 \cdot 15 = 60$. We need two integers whose product is 60, and whose sum is 16. The obvious two are $M = 10$ and $N = 6$, so we have from equation (5):

$$4x^2 + 16x + 15 = \frac{(4x+10)(4x+6)}{4} = \frac{2(2x+5) \cdot 2(2x+3)}{4} = (2x+5)(2x+3).$$

Here, a common factor of 2 is factored from each binomial factor in the numerator, and their product cancels the 4 in the denominator.

4. $12x^2 + 11x + 2$.

Here, $a = 12$, $b = 11$, $c = 2$. So $ac = 12 \cdot 2 = 24$. We need two integers whose product is 24, and whose sum is 11. These are 3 and 8, so we have from equation (5)

$$12x^2 + 11x + 2 = \frac{(12x+3)(12x+8)}{12} = \frac{3(4x+1) \cdot 4(3x+2)}{12} = (4x+1)(3x+2).$$

The first binomial factor has a common factor of 3 that can be factored out, while the second has a common factor of 4 that can be factored out. The product of these two factors, which is $3 \cdot 4 = 12$, will cancel the 12 in the denominator to yield the final result.

5. $3x^2 - 7x + 2$.

We have $a = 3$, $b = -7$, $c = 2$. So $ac = 3 \cdot 2 = 6$. We need two integers whose product is 6, and whose sum is -7. These are -6 and -1, so we have from equation (5)

$$3x^2 - 7x + 2 = \frac{(3x-6)(3x-1)}{3} = \frac{3(x-2)(3x-1)}{3} = (x-2)(3x-1).$$

The first binomial factor has a common factor of 3 that can be factored out, and this 3 will cancel the 3 in the denominator.