Radius of a Circle Inscribed in a Triangle

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We have a circle of radius r inscribed in an arbitrary triangle with sides of lengths a, b, and c, as shown in the following Figure:

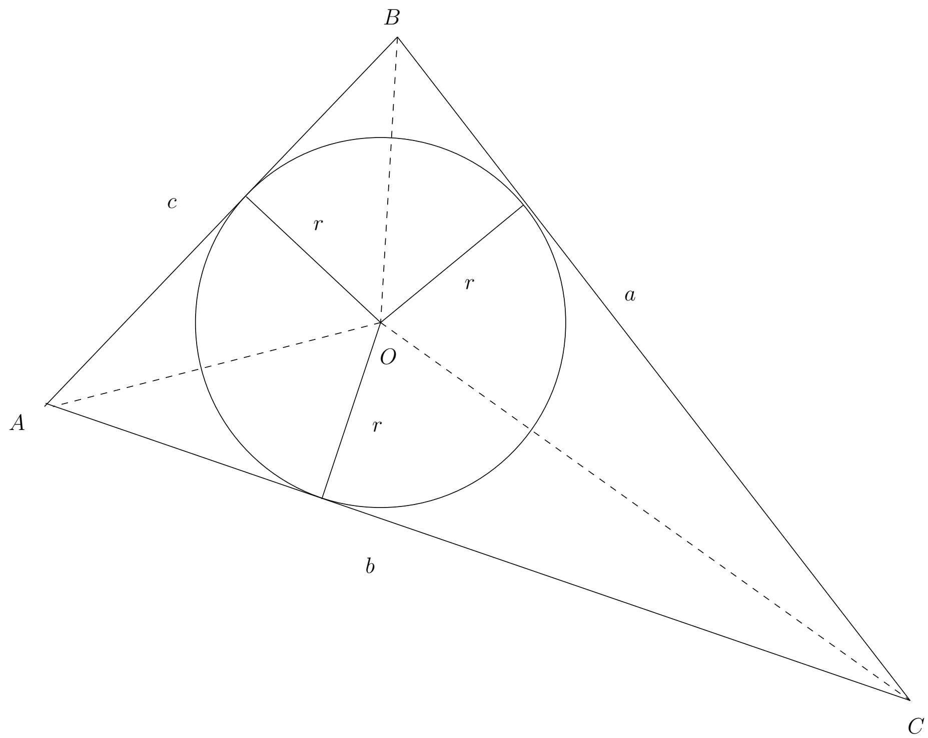

Figure 1: Circle of radius r inscribed in a triangle with sides a, b, and c.

The dotted lines from the center of the circle to each vertex of the triangle create three triangles, namely, $\triangle ABO$, $\triangle BCO$, and $\triangle CAO$, each of which has altitude r, the radius of the circle. The total area A_{abc} of $\triangle ABC$ is the sum of the areas of the three triangles $\triangle ABO$, $\triangle BCO$, and $\triangle CAO$:

$$A_{abc} = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}r(a + b + c), \text{ after factoring the common factor of } r.$$

In this equation, the expression $\frac{1}{2}(a+b+c)$, that is, half the perimeter $(a+b+c)$ of $\triangle ABC$, is called the *semi-perimeter* of $\triangle ABC$, usually denoted by s. Replacing $\frac{1}{2}(a+b+c)$ in the area formula by s, we have simply

$$A_{abc} = rs.$$

From this expression we obtain a simple formula for the radius r of the inscribed circle in terms of the area of $\triangle ABC$:

$$r = \frac{A_{abc}}{s}. \tag{1}$$

We will show later that the area of the triangle is given in terms of its semi-perimeter s, and sides a, b, and c, by *Heron's formula*:

$$A_{abc} = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

Using this formula in place of A_{abc} in equation (1) yields the desired expression of the radius of the inscribed circle in terms of s and the three sides of the triangle in which the circle is inscribed:

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad (2)$$

A Derivation of Heron's Formula for the Area of a Triangle

The most difficult part of a complete demonstration of equation (2) is that of deriving Heron's formula. We base our derivation on the following Figure:

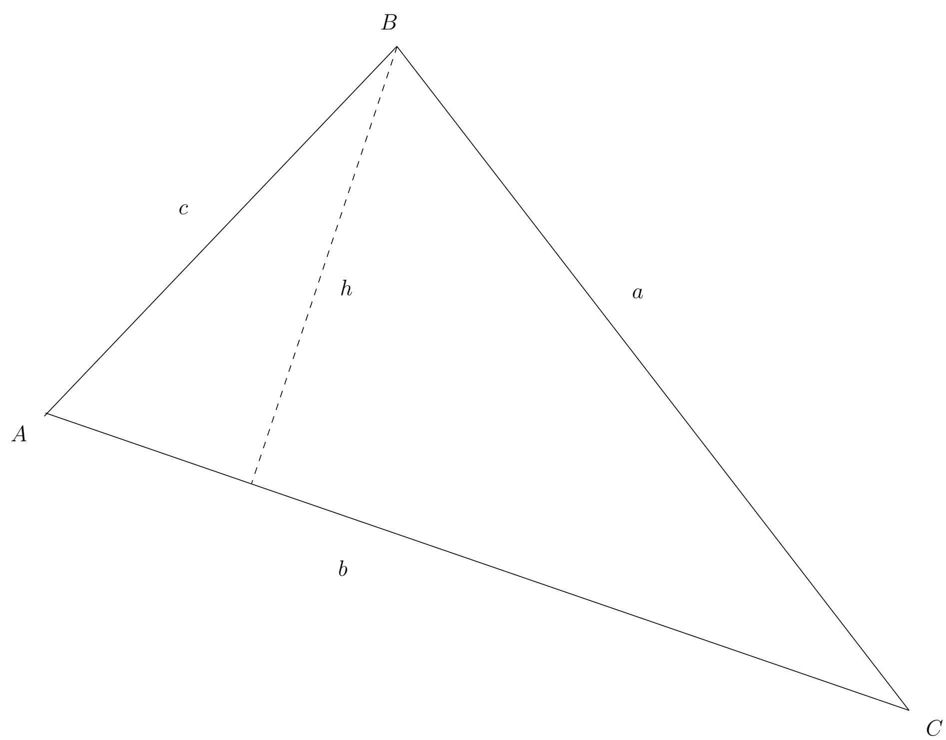

Figure 2: Triangle of sides a, b, and c, and altitude h.

The area of $\triangle ABC$ is just

$$A_{abc} = \frac{1}{2}bh,$$

and from the right triangle with hypotenuse c and side h we have

$$\sin A = \frac{h}{c},$$

from which $h = c\sin A$. Substituting this expression for h in the area formula gives us

$$A_{abc} = \frac{1}{2}bc\sin A.$$

The square of this area is then

$$A_{abc}^2 = \frac{1}{4}b^2c^2\sin^2 A. \qquad (3)$$

Applying the law of cosines to the included angle A of $\triangle ABC$, we find

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

from which

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

hence

$$\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{4b^2c^2 - \left(b^2 + c^2 - a^2\right)^2}{4b^2c^2},$$

and this gives us

$$b^2c^2 \sin^2 A = \frac{1}{4}\left[4b^2c^2 - \left(b^2 + c^2 - a^2\right)^2\right].$$

Substituting this expression for $b^2c^2 \sin^2 A$ in equation (3) yields for the square of the area:

$$A_{abc}^2 = \frac{1}{16}\left[4b^2c^2 - \left(b^2 + c^2 - a^2\right)^2\right]. \tag{4}$$

Since the difference of two squares factors as $x^2 - y^2 = (x + y)(x - y)$, the right-hand side of equation (4) can be factored and rewritten as

$$\begin{aligned} A_{abc}^2 &= \frac{1}{16}\left[2bc + \left(b^2 + c^2 - a^2\right)\right] \cdot \left[2bc - \left(b^2 + c^2 - a^2\right)\right], \\ &= \frac{1}{16}\left[(b^2 + 2bc + c^2) - a^2\right] \cdot \left[a^2 - \left(b^2 - 2bc + c^2\right)\right], \text{ after regrouping,} \\ &= \frac{1}{16}\left[(b + c)^2 - a^2\right] \cdot \left[a^2 - \left(b - c\right)^2\right], \end{aligned}$$

after factoring the perfect square trinomials in parentheses. Again, factoring the differences of squares in each bracketed product yields

$$A_{abc}^2 = \frac{1}{16}(b + c + a)(b + c - a)(a + b - c)(a + c - b). \tag{5}$$

Recalling that the semi-perimeter $s = \frac{1}{2}(a + b + c)$, from which

$$2s = a + b + c,$$

we obtain expressions for the sums of any two sides of the triangle:

$$a + b = 2s - c, \qquad b + c = 2s - a, \qquad \text{and} \qquad a + c = 2s - b.$$

These can be used to replace the same sums, occurring in equation (5), in terms of s. By doing so, we obtain the following formula for the area:

$$A_{abc}^2 = \frac{1}{16}(2s)(2s - 2a)(2s - 2c)(2s - 2b).$$

Factoring the common factors of 2 and re-ordering the products, this simplifies to

$$A_{abc}^2 = \frac{1}{16}\left[16s(s - a)(s - b)(s - c)\right] = s(s - a)(s - b)(s - c),$$

and taking the square root of both sides yields, finally, Heron's formula:

$$A_{abc} = \sqrt{s(s - a)(s - b)(s - c)}. \tag{6}$$