# A Note on Euler's Formula

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## 1 Euler's Formula

The general complex exponential function $e^z$, where $z$ is any *complex* number of the form $(a+ib)$, has been called "the most important function in mathematics" by the author of a highly regarded advanced mathematics text [1, Prologue]. Euler's formula relates the *special* complex exponential function $e^{i\theta}$ to the familiar trigonometric functions $\cos\theta$ and $\sin\theta$. It is simple to state, yet profound in its consequences:

$$e^{i\theta} \;=\; \cos\theta + i\sin\theta. \tag{1}$$

It can be derived (in a mathematically non-rigorous, that is, *formal* manner) using ideas from a first course in calculus. For those readers who have not taken a calculus course, or otherwise have no interest in such a derivation, please skip ahead to the next Section on applications of the formula.

To derive the relation, consider the complex function

$$u(\theta) \;=\; \cos\theta + i\sin\theta. \tag{2}$$

Take the derivative of both sides with respect to $\theta$ to obtain

$$u'(\theta) \;=\; -\sin\theta + i\cos\theta.$$

Now, notice that $i^2 = -1$, so the right-hand side of this result can be written as

$$u'(\theta) \;=\; i^2\sin\theta + i\cos\theta,$$

or, after factoring out the common factor of $i$ and rearranging:

$$u'(\theta) \;=\; i\,(\cos\theta + i\sin\theta).$$

But on the right-hand side, the factor multiplying $i$ is just the original function $u(\theta)$, that is,

$$u'(\theta) \;=\; i\,u(\theta). \tag{3}$$

Dividing both sides by $u(\theta)$, and recalling that

$$\frac{u'(\theta)}{u(\theta)} \;=\; \frac{d}{d\theta}\left\{\ln\left[u(\theta)\right]\right\},$$

it follows from (3) that

$$\frac{d}{d\theta}\left\{\ln\left[u(\theta)\right]\right\} \;=\; i,$$

which is easily integrated with respect to $\theta$ to get the indefinite integral

$$\ln\left[u(\theta)\right] \;=\; i\theta + C_0,$$

where $C_0$ is an arbitrary constant. Rewriting this in exponential form yields

$$u(\theta) \;=\; e^{i\theta + C_0} \;=\; e^{i\theta}\, e^{C_0} \;=\; C\, e^{i\theta}.$$

where $C = e^{C_0}$ is another arbitrary constant. But from the definition (2), we see that we must have $u(0) = \cos 0 + i \sin 0 = 1$, so from the last equation $u(0) = Ce^{i0} = C = 1$ since $e^0 = 1$, leaving

$$u(\theta) \;=\; e^{i\theta} \;=\; \cos\theta + i\sin\theta, \tag{4}$$

where the second equality results from definition (2), thus completing a formal derivation of Euler's formula (1).

## 2 Elementary Applications

Suppose you want the trigonometric identities for the cosine and sine of the sum of two angles: $\cos(x+y)$ and $\sin(x+y)$. First, we know that they are the real and imaginary parts, respectively, of $e^{i(x+y)}$, that is,

$$e^{i(x+y)} \;=\; \cos(x+y) + i\sin(x+y). \tag{5}$$

But notice that this exponential can be written alternatively as a *product* of exponentials, namely

$$e^{i(x+y)} \;=\; e^{ix}\, e^{iy}. \tag{6}$$

Using Euler's formula (1) for the exponentials on the right-hand side of (6), we have for their product:

$$\begin{aligned} e^{i(x+y)} \;&=\; (\cos x + i\sin x)(\cos y + i\sin y), \\ &=\; \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y, \\ &=\; \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y). \end{aligned} \tag{7}$$

Comparing (7) to (5), the real part of the right-hand side must equal $\cos(x+y)$, while the imaginary part must equal $\sin(x+y)$, giving us the usual formulas:

$$\cos(x+y) \;=\; \cos x\cos y - \sin x\sin y, \quad \text{and} \quad \sin(x+y) \;=\; \sin x\cos y + \cos x\sin y. \tag{8}$$

Replacing $y$ by $-y$ in each case yields the appropriate relations for the sine and cosine of the difference of two angles, recalling that the cosine is even, $\cos(-y) = \cos y$ and the sine is odd, $\sin(-y) = -\sin y$ (causing a change in sign of the second term in each case):

$$\cos(x-y) \;=\; \cos x\cos y + \sin x\sin y, \quad \text{and} \quad \sin(x-y) \;=\; \sin x\cos y - \cos x\sin y. \tag{9}$$

These formulas are worth their weight in gold! From them follow many other identities, as we now show.

For example, letting $x = y = \theta$ in these two expressions gives the double-angle formulas:

$$\cos(2\theta) \;=\; \cos^2\theta - \sin^2\theta, \quad \text{and} \quad \sin(2\theta) \;=\; 2\sin\theta\cos\theta. \tag{10}$$

Letting $x = y = \theta/2$ gives expressions involving half-angles, namely,

$$\cos\theta \;=\; \cos^2(\theta/2) - \sin^2(\theta/2), \quad \text{and} \quad \sin\theta \;=\; 2\sin(\theta/2)\cos(\theta/2). \tag{11}$$

Substituting $\sin^2(\theta/2) = 1 - \cos^2(\theta/2)$ in the first of equations (11) yields after a little algebra:

$$\cos^2(\theta/2) \;=\; \frac{1+\cos\theta}{2},$$

or

$$\cos(\theta/2) \;=\; \pm\sqrt{\frac{1+\cos\theta}{2}}\,. \tag{12}$$

Substituting instead $\cos^2(\theta/2) = 1 - \sin^2(\theta/2)$ in the first of equations (11) yields

$$\sin^2(\theta/2) \;=\; \frac{1-\cos\theta}{2},$$

or

$$\sin(\theta/2) \;=\; \pm\sqrt{\frac{1-\cos\theta}{2}}\,. \tag{13}$$

Returning to equations (8), we can use the first to obtain

$$\cos(x+y)\;+\;\cos(x-y)\;=\;\cos x\cos y-\cancel{\sin x\sin y}+\cos x\cos y+\cancel{\sin x\sin y}\;=\;2\cos x\cos y,$$

where $y$ was replaced by $-y$ to obtain the first equality. In this equation, set $\alpha = x+y$, and $\beta = x-y$. These two linear equations can be solved for $x$ and $y$ in terms of $\alpha$ and $\beta$ to obtain $x = (\alpha+\beta)/2$, and $y = (\alpha-\beta)/2$, and making these substitutions in the last equation leads to another useful formula:

$$\cos\alpha\;+\;\cos\beta\;=\;2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]. \tag{14}$$

If we instead use the first of equations (8) to subtract the cosines of two angles, we get

$$\cos(x+y)\;-\;\cos(x-y)\;=\;\cancel{\cos x\cos y}-\sin x\sin y-\cancel{\cos x\cos y}-\sin x\sin y\;=\;-2\sin x\sin y,$$

and again introducing $\alpha = x+y$, and $\beta = x-y$, solving for $x$ and $y$, and substituting appropriately, we obtain

$$\cos\alpha\;-\;\cos\beta\;=\;-2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]. \tag{15}$$

Similarly, the second equation of (8) can be used to obtain

$$\sin(x+y)\;+\;\sin(x-y)\;=\;\sin x\cos y+\cancel{\cos x\sin y}+\sin x\cos y-\cancel{\cos x\sin y}\;=\;2\sin x\cos y.$$

Again introducing $\alpha = x+y$, and $\beta = x-y$, solving them for $x$ and $y$, and substituting into the last equation yields

$$\sin\alpha\;+\;\sin\beta\;=\;2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2], \tag{16}$$

in agreement with the formula given, for example, in Appendix E of the physics textbook [2], where it is used in Section 16-10 on wave interference to write the superposition (sum) of two *traveling waves* in a more useful form. For completeness, replacement of $\beta$ by $-\beta$ in (16) yields the difference of sines:

$$\sin\alpha\;-\;\sin\beta\;=\;2\sin[(\alpha-\beta)/2]\cos[(\alpha+\beta)/2]. \tag{17}$$

It is perhaps worthwhile to gather together the last four formulas for the sums and differences of the cosines and sines of two angles:

$$\cos\alpha\;+\;\cos\beta\;=\;\;\;2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2], \tag{18}$$

$$\cos\alpha\;-\;\cos\beta\;=\;-2\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2], \tag{19}$$

$$\sin\alpha\;+\;\sin\beta\;=\;\;\;2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2], \tag{20}$$

$$\sin\alpha\;-\;\sin\beta\;=\;\;\;2\sin[(\alpha-\beta)/2]\cos[(\alpha+\beta)/2]. \tag{21}$$

Notice that if we divide equation (20) by (18) we obtain the interesting result that

$$\frac{\sin\alpha\;+\;\sin\beta}{\cos\alpha\;+\;\cos\beta}\;=\;\tan[(\alpha+\beta)/2]. \tag{22}$$

Replacing $\beta$ by $-\beta$ (or dividing equation (21) by (18)) then yields

$$\frac{\sin\alpha\;-\;\sin\beta}{\cos\alpha\;+\;\cos\beta}\;=\;\tan[(\alpha-\beta)/2]. \tag{23}$$

Returning to Euler's formula, the following problem recently surfaced in a trigonometry course: students were asked in a computer-generated problem to "simplify" $16\sin^8 x$. Our approach will involve an application of the binomial theorem, so we have listed the binomial coefficients up to $n = 9$ in Pascal's triangle, illustrated in Figure 1. Begin by replacing $\sin x$ by its complex representation (29) to obtain

$$16\,\sin^8 x\;=\;16\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^8.$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

Figure 1: Pascal's Triangle

Now apply the binomial theorem for $n = 8$, using the second from the last row of Pascal's triangle:

$$\begin{aligned} 16\,\sin^8 x \;&=\; \frac{16}{(2i)^8}\left(e^{ix} - e^{-ix}\right)^8, \\ &= \frac{16}{256}\big(\,e^{8ix} - 8\,e^{7ix}\,e^{-ix} + 28\,e^{6ix}\,e^{-2ix} - 56\,e^{5ix}\,e^{-3ix} + 70\,e^{4ix}\,e^{-4ix} \\ &\qquad - 56\,e^{3ix}\,e^{-5ix} + 28\,e^{2ix}\,e^{-6ix} - 8\,e^{ix}\,e^{-7ix} + e^{-8ix}\,\big), \\ &= \frac{1}{16}\big(\,e^{8ix} - 8\,e^{6ix} + 28\,e^{4ix} - 56\,e^{2ix} + 70 - 56\,e^{-2ix} + 28\,e^{-4ix} - 8\,e^{-6ix} + e^{-8ix}\,\big) \\ &= \frac{1}{16}\big[\,\left(e^{8ix} + e^{-8ix}\right) - 8\left(e^{6ix} + e^{-6ix}\right) + 28\left(e^{4ix} + e^{-4ix}\right) - 56\left(e^{2ix} + e^{-2ix}\right) + 70\,\big]. \end{aligned}$$

In the last equation, we recognize from the complex exponential representation (29) of the cosine function that for any integer $k$: $\left(e^{kix} + e^{-kix}\right) \;=\; 2\cos(kx)$, so we have:

$$16\,\sin^8 x \;=\; \frac{1}{16}\big[\,2\,\cos(8x) - 8\cdot 2\,\cos(6x) + 28\cdot 2\,\cos(4x) - 56\cdot 2\,\cos(2x) + 70\,\big],$$

or, factoring out a common factor of 2:

$$16\,\sin^8 x \;=\; \frac{1}{8}\big[\,\cos(8x) - 8\,\cos(6x) + 28\,\cos(4x) - 56\,\cos(2x) + 35\,\big], \tag{24}$$

for the final result. This form of the answer contains no *products* of cosine functions. The actual form of the answer given by the computer module, however, did not include the $\cos(6x)$ term. To obtain the computer's form of the answer from ours requires considerable manipulation of this term:

$$\begin{aligned} \cos(6x) \;&=\; \cos(2x + 4x) \;=\; \cos(2x)\cos(4x) \;-\; \sin(2x)\sin(4x), \\ &= \cos(2x)\cos(4x) \;-\; \sin(2x)\big[2\,\sin(2x)\cos(2x)\big], \\ &= \cos(2x)\cos(4x) \;-\; 2\sin^2(2x)\;\cos(2x), \\ &= \cos(2x)\cos(4x) \;-\; 2\left[1 \;-\; \cos^2(2x)\right]\;\cos(2x), \\ &= \cos(2x)\cos(4x) \;-\; 2\,\cos(2x) \;+\; 2\,\cos^2(2x)\;\cos(2x), \\ &= \cos(2x)\cos(4x) \;-\; 2\,\cos(2x) \;+\; \cancel{2}\left[\frac{1 \;+\; \cos(4x)}{\cancel{2}}\right]\cos(2x), \\ &= 2\cos(2x)\cos(4x) \;-\; \cos(2x). \end{aligned} \tag{25}$$

Substituting this expression for $\cos(6x)$ in our result (24) yields

$$16\,\sin^8 x \;=\; \frac{1}{8}\Big\{\;\cos(8x) - 8\,\big[\,2\cos(2x)\cos(4x) \;-\; \cos(2x)\,\big] + 28\,\cos(4x) - 56\,\cos(2x) + 35\,\Big\},$$

which reduces after collecting like terms to:

$$16\,\sin^8 x \;=\; \frac{1}{8}\big[\cos(8x) - 16\,\cos(2x)\cdot\cos(4x) + 28\,\cos(4x) - 48\,\cos(2x) + 35\,\big], \tag{26}$$

the form of the answer given by the computer module, which exchanges the $\cos(6x)$ term in (24) for a term containing a *product* of cosine terms.

If, in Euler's formula (1), either $i$ is replaced by $-i$, or $\theta$ is replaced by $-\theta$, then since $\cos(-\theta) = \cos\theta$, and $\sin(-\theta) = -\sin\theta$, we obtain by either method:

$$e^{-i\theta} \;=\; \cos\theta - i\sin\theta. \tag{27}$$

The sum and difference of (1) and (27) yield:

$$e^{i\theta} + e^{-i\theta} \;=\; 2\cos\theta, \quad \text{and} \quad e^{i\theta} - e^{-i\theta} \;=\; 2i\sin\theta, \tag{28}$$

respectively, from which we get

$$\cos\theta \;=\; \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \text{and} \quad \sin\theta \;=\; \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{29}$$

respectively. These formulas are useful for finding indefinite integrals of the trigonometric functions.

## References

[1] W. Rudin, *Real and Complex Analysis*, McGraw–Hill, New York, NY, 2nd ed., 1974.

[2] D. Halliday, R. Resnick, and J. Walker, *Principles of Physics Extended*, Wiley, 9th ed., 2010.