Arithmetic and Geometric Sequences and Series, and the Binomial Theorem

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Arithmetic Sequences and Series

An *arithmetic sequence* (or arithmetic progression) is a sequence in which the *difference between any two successive terms* is a constant value, d, called the *common difference*. That is,

$$a_{i+1} - a_i = d, \qquad i \geq 1. \tag{1}$$

Thus, given the first term a_1, all other terms are defined recursively by

$$a_{i+1} = a_i + d, \qquad i \geq 1. \tag{2}$$

Letting i take on successive values greater than or equal to 1, we find

$$a_2 = a_1 + d,$$

$$a_3 = a_2 + d = \underbrace{a_1 + d} + d = a_1 + d + d = a_1 + 2d,$$

$$a_4 = a_3 + d = \underbrace{a_1 + 2d} + d = a_1 + 3d,$$

and in general, the nth term is given by

$$a_n = a_1 + (n-1)d. \tag{3}$$

If you are given the first term a_1 and the spacing d, then you can calculate any other term of the sequence using equation (3). If, on the other hand, you are given *any two terms*, say the kth and the mth (where k and m can be any two distinct natural numbers ≥ 1), then you can determine the common difference d as follows. Applying equation (3) to the kth and the mth terms, we have

$$a_k = a_1 + (k-1)d,$$

and

$$a_m = a_1 + (m-1)d.$$

One of the two values, k or m, is greater, so assume that m is the greater of the two. Then subtract a_k from a_m to obtain

$$\begin{aligned} a_m - a_k &= a_1 + (m-1)d - [\,a_1 + (k-1)d\,], \\ &= (m-k)d, \end{aligned}$$

from which the spacing is given by

$$d = \frac{a_m - a_k}{m - k}. \qquad (4)$$

Once d has been determined from this expression, the first term a_1 can be found from the equation for either a_k or a_m:

$$a_1 = a_k - (k-1)d = a_m - (m-1)d. \qquad (5)$$

The *sum* of the first n terms of an arithmetic sequence is called a finite *arithmetic series*. This sum can be shown to be given by

$$S_n = \sum_{k=1}^{n} a_k = \frac{n(a_n + a_1)}{2} = \frac{n[(n-1)d + 2a_1]}{2}, \qquad (6)$$

where the last equality results from using the definition (3) to replace $a_n = a_1 + (n-1)d$ in the previous equality.

Geometric Sequences and Series

A *geometric sequence* (or geometric progression) is a sequence in which the *ratio of any two successive terms* is a constant value, r, called the *common ratio*. That is,

$$\frac{a_{i+1}}{a_i} = r, \qquad i \geq 1. \qquad (7)$$

Thus, given the first term a_1, all other terms are defined recursively by

$$a_{i+1} = a_i r. \qquad (8)$$

Letting i take on successive values greater than or equal to 1, we find

$$a_2 = a_1 r,$$

$$a_3 = a_2 r = a_1 r \cdot r = a_1 r^2,$$

$$a_4 = a_3 r = a_1 r^3,$$

and in general, the nth term is given by

$$a_n = a_1 r^{n-1}. \qquad (9)$$

If you are given the first term a_1 and the ratio r, then you can calculate any other term of the sequence using equation (9). If, on the other hand, you are given *any two terms*, say the kth and the mth (where k and m can be any two distinct natural numbers ≥ 1), then you can determine the common ratio r as follows. Applying equation (9) to the kth and the mth terms, we have

$$a_k = a_1 r^{k-1},$$

and

$$a_m = a_1 r^{m-1}.$$

Assuming $m > k$, divide a_m by a_k to obtain

$$\frac{a_m}{a_k} = \frac{a_1 r^{m-1}}{a_1 r^{k-1}} = r^{m-k},$$

then by raising both sides of this expression to the $1/(m-k)$ power, that is, by taking the $(m-k)$th *root* of the ratio a_m/a_k, we will find the value of the common ratio r:

$$r = \left(\frac{a_m}{a_k}\right)^{1/(m-k)} = \sqrt[(m-k)]{\frac{a_m}{a_k}}. \tag{10}$$

Once the common ratio has been determined, the first term a_1 can be found from either the equation for a_k, or the equation for a_m:

$$a_1 = \frac{a_k}{r^{k-1}} = \frac{a_m}{r^{m-1}}. \tag{11}$$

The *sum* of the first n terms of a geometric sequence is called a finite *geometric series*. This sum can be shown to be given by

$$S_n = \sum_{k=1}^{n} a_k = \frac{a_1 - a_1 r^n}{1-r} = a_1\left(\frac{1-r^n}{1-r}\right) = \frac{a_1 - a_n r}{1-r}, \tag{12}$$

where the last equality results from using equation (9) to replace $a_1 r^n = a_n r$ in the first equality.

If $|r| < 1$, then as the number of terms increases without bound (in which case the finite series becomes an infinite series) the *sum of the infinite geometric series* is (more precisely, *converges to*)

$$S = \sum_{k=1}^{\infty} a_k = \frac{a_1}{1-r}, \text{ for an infinite geometric series when } |r| < 1. \tag{13}$$

If $|r| > 1$, the infinite geometric series does *not* converge to a finite value (the sum increases without bound, or diverges).

The Binomial Theorem

The binomial theorem gives the following result for the expansion of a general binomial expression, $(ax + by)$, raised to the nth power:

$$(ax + by)^n = \sum_{r=0}^{n} {}_nC_r \, (ax)^{n-r} \, (by)^r, \tag{14}$$

where the "binomial coefficient" ${}_nC_r = \binom{n}{r}$ is the number of combinations of n things taken r at a time ("n choose r"), computed by the formula

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)! \, r!}. \tag{15}$$

The binomial coefficients ${}_nC_r = \binom{n}{r}$ can also be determined by constructing Pascal's triangle, shown in Figure 1 for the first 10 values of n from $n = 0$ to $n = 9$.

$$
\begin{array}{c}
1 \quad \ldots \quad n=0 \\
1 \quad 1 \quad \ldots \quad n=1 \\
1 \quad 2 \quad 1 \quad \ldots \quad n=2 \\
1 \quad 3 \quad 3 \quad 1 \quad \ldots \quad n=3 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \quad \ldots \quad n=4 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \quad \ldots \quad n=5 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \quad \ldots \quad n=6 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \quad \ldots \quad n=7 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \quad \ldots \quad n=8 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \quad \ldots \quad n=9
\end{array}
.
$$

Figure 1: Pascal's Triangle for $n = 0$ to 9

The sum (14) defining a binomial expansion is referred to as a binomial *series*. It is convenient to write it as

$$(ax + by)^n = \sum_{r=0}^{n} b_r = \sum_{r=0}^{n} B_r \cdot x^{n-r} y^r, \tag{16}$$

where we have denoted the terms of the series by

$$b_r = {}_nC_r \, (ax)^{n-r} \, (by)^r = {}_nC_r \, a^{n-r} \, b^r \cdot x^{n-r} y^r = B_r \cdot x^{n-r} y^r, \tag{17}$$

and where the coefficient of the term containing $x^{n-r}y^r$ has been denoted by

$$B_r \;=\; {}_nC_r\, a^{n-r}\, b^r. \tag{18}$$

Notice that the sum (14) defining a binomial expansion begins with $r = 0$, hence $r = 0$ is the first term, $r = 1$ is the second term, $r = 2$ is the third term and, in general, $r = k - 1$ is the kth *term* of the series. Thus, for example, the fourth term ($k = 4$) of the series for any value of n corresponds to $r = 3$ in the sum, hence the fourth term is

$$b_3 \;=\; {}_nC_3\,(ax)^{n-3}\,(by)^3 \;=\; {}_nC_3\, a^{n-3}\, b^3 \cdot x^{n-3}y^3. \tag{19}$$

As an example, the binomial series for $(2x + 5y)^4$ is

$$\begin{aligned}(2x+5y)^4 \;=\; \sum_{r=0}^{4} {}_4C_r(2x)^{4-r}(5y)^r \;&=\; {}_4C_0(2x)^{4-0}(5y)^0 + {}_4C_1(2x)^{4-1}(5y)^1 \\ &+ {}_4C_2(2x)^{4-2}(5y)^2 + {}_4C_3(2x)^{4-3}(5y)^3 + {}_4C_4(2x)^{4-4}(5y)^4,\end{aligned}$$

which simplifies to

$$\begin{aligned}(2x+5y)^4 \;&=\; {}_4C_0(2x)^4 + {}_4C_1(2x)^3(5y) + {}_4C_2(2x)^2(5y)^2 + {}_4C_3(2x)(5y)^3 + {}_4C_4(5y)^4 \\ &=\; {}_4C_0(16x^4) + {}_4C_1(8x^3)(5y) + {}_4C_2(4x^2)(25y^2) + {}_4C_3(2x)(125y^3) + {}_4C_4(625y^4).\end{aligned}$$

The binomial coefficients can be either calculated, or found from the $n = 4$ row of Pascal's triangle: ${}_4C_0 = 1,\ {}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4,$ and ${}_4C_4 = 1,$ hence

$$\begin{aligned}(2x+5y)^4 \;&=\; 1\cdot 16x^4 + 4\cdot(8x^3)(5y) + 6\cdot(4x^2)(25y^2) + 4\cdot(2x)(125y^3) + 1\cdot 625y^4, \\ &=\; 16x^4 + 160x^3y + 600x^2y^2 + 1000xy^3 + 625y^4. \qquad (20)\end{aligned}$$

Suppose, however, that you have not computed the complete series as we have just done, and you are asked to find, say, the *third term* of the expansion of $(2x + 5y)^4$. This corresponds to $n = 4$ and $r = 2$ of the binomial series (remember that for the $k = 3$ term, the value of r is $r = k - 1 = 2$, always *one less* than the term number k), so it is given according to (17) by

$$b_2 \;=\; {}_4C_2\,(2x)^{4-2}\,(5y)^2 \;=\; 6\cdot 2^2\cdot 5^2\cdot x^2y^2 = 6\cdot 4\cdot 25\cdot x^2y^2 = 600x^2y^2,$$

in agreement with the expansion (20). On the other hand, if you are asked to find, for example, the *coefficient* of the x^3y term of this series, that is, the coefficient of the term for which $r = 1$ (since r is determined by the exponent of y in our binomial series), it is given according to (17) by

$$B_1 \;=\; {}_4C_1\, a^{4-1}\, b^1 \;=\; {}_4C_1\, a^3\, b \;=\; 4\cdot 2^3\cdot 5^1 \;=\; 4\cdot 8\cdot 5 \;=\; 160,$$

again in agreement with the expansion (20).